A-INFINITY STRUCTURES RELATED TO BI-KOSZUL ALGEBRAS

JUN-RU SI AND DI-MING LU

Abstract. Let A be a bi-Koszul algebra, we describe all possible A_∞-algebra structures on the Ext-algebra $E(A)$, and prove that $E(A)$ must be $[m_2, m_3]$-finitely generated. An equivalent description for a connected graded algebra to be a bi-Koszul algebra is given in terms of A_∞-language. The case that $E(A)$ is endowed with minimal number of multiplications is discussed for decomposition.

Introduction

To understand certain homological properties of graded algebras whose trivial modules admit non-pure resolutions, the authors introduced what they have called bi-Koszul algebras in [10]. Any non-Koszul Artin-Schelter regular algebras generated in degree 1 of global dimension four are the examples. Different from algebras with certain pure resolutions of the trivial modules (such as Koszul algebras [12], d-Koszul algebras [2], piecewise-Koszul algebras [11], *etc.*) and $\mathcal{K}_2$-algebras [3], bi-Koszul algebras lose a nice homological property that their Ext-algebras are finitely generated.

This may be remodeled if one endows "generating" with an appropriate meaning. For example, Keller claimed that Ext-algebras are A_∞-generated by their homogenous components of degree 1 for a large number of graded algebras [6, Proposition 1(b)]. Though it is nice to have finite generating components on one hand, it maybe require, as a redeem, infinite multiplications to guarantee finitely generating on the other hand. One of goals of this paper is to find out the multiplications on a given Ext-algebra as less as possible to carry out the finitely generating. We prove that the Ext-algebra of any bi-Koszul algebra is $[m_2, m_3]$-finitely generated.

We examine all possible A_∞-algebra structures, corresponding to a bi-Koszul algebra A determined by Δ_d, to get at most five multiplications $m_2, m_3, m_4, m_d, m_{d+1}$. An A_∞-version duality theory for a bi-Koszul algebra is given. The case that $E(A)$ is endowed with minimal number of multiplications m_2, m_d, m_{d+1} is especially interesting for decomposition. Two single A_∞-algebras are obtained here and can be returned to the A_∞-structure of $E(A)$ by a bridge.

2000 *Mathematics Subject Classification.* 16E05, 16E40, 16S37, 16W50.

Key words and phrases. Ext-algebra, A_∞-algebra, bi-Koszul algebra.

The work was supported by the NSFC (Grant No. 10571152) and partially by the NSF of Zhejiang Province of China (Grant No. J20080154).

We introduce a modified concept of "generating" which reflects some balance between multiplications and elements in the A_∞-algebra system and prove that there exists an A_∞-algebra structure on $E(A)$ of a bi-Koszul algebra A such that $E(A)$ is $[m_2, m_3]$-finitely generated by $E^1(A)$, $E^2(A)$ and $E^3(A)$. A new criterion for a bi-Koszul algebra to be strongly is given in the A_∞-version. Based on the fact that the A_∞-Ext-algebra $E(A)$ is unique up to quasi-isomorphism, we discuss whether an A_∞-algebra $E(A)$ is generated by $E^1(A)$.

1. A_∞-algebras and bi-Koszul algebras

In this section, we review basic material necessary for the paper: Ext-algebras, A_∞-algebras, and bi-Koszul algebras.

1.1. **Ext-algebras.** Throughout we fix a field $\mathbb{F}$. We always assume that a graded algebra $A = \mathbb{F} \oplus A_1 \oplus A_2 \oplus \cdots$ is locally finite, connected, and generated in degree 1. The graded Jacobson radical of A, denoted by J, is $J = A_{\geq 1}$. Let $Gr(A)$ denote the category of graded left A-modules. The morphisms in this category, denoted by $\mathrm{Hom}_{Gr(A)}(M, N)$ for $M, N \in Gr(A)$, are graded A-module maps of degree zero. For $M \in Gr(A)$, we denote the n^{th} *shift* of M by $M[n]$ where $M[n]_j = M_{j+n}$.

We write $\underline{\mathrm{Ext}}_A^*$ the derived functor of the graded $\underline{\mathrm{Hom}}_A^*$ functor

$$\underline{\mathrm{Hom}}_A^*(M, N) := \bigoplus_n \mathrm{Hom}_{Gr(A)}(M, N[n]),$$

and denote

$$E(A) := \underline{\mathrm{Ext}}_A^*(\mathbb{F}, \mathbb{F}), \qquad \mathsf{E}(M) := \underline{\mathrm{Ext}}_A^*(M, \mathbb{F}),$$

the *Koszul dual* of the algebra A and the *Koszul dual* of the module $M \in Gr(A)$, respectively. $E(A)$ is equipped with a bigraded algebra structure by the Yoneda product with the $(i,j)^{\text{th}}$ component $\underline{\mathrm{Ext}}_A^i(\mathbb{F}, \mathbb{F})_{-j}$, we also call it (classical) *Ext-algebra* of A. Here, i is the *cohomology degree* and $-j$ is the *internal degree*. Note that the internal degree in $E(A)$ is non-positive. For simplicity, we promise $E_j^i(A) := \underline{\mathrm{Ext}}_A^i(\mathbb{F}, \mathbb{F})_{-j}$. Similarly, $\mathsf{E}(M)$ is a bigraded left $E(A)$-module with the $(i,j)^{\text{th}}$ component $\mathsf{E}_j^i(M) := \underline{\mathrm{Ext}}_A^i(M, \mathbb{F})_{-j}$.

The (classical) Ext-algebra $E(A)$ carries rich information about the algebra A and its module category, but it does not contain enough information to recover the original algebra in general, the "hidden" information is revealed in the A_∞-world.

1.2. A_∞**-algebras.** There are different methods to give the definition of an A_∞-algebra (algebraical, geometrical, operadic, etc.), but here we prefer the algebraical definition of an A_∞-algebra. We refer to [7] or [9] for the details.

Definition 1.1. An A_∞*-algebra* over a field $\mathbb{F}$ is a $\mathbb{Z}$-graded vector space

$$E = \bigoplus_{p \in \mathbb{Z}} E^p$$

endowed with a family of graded $\mathbb{F}$-linear maps

$$m_n : E^{\otimes n} \to E, \ \ (n \geq 1)$$

of degree $2-n$ satisfying the *Stasheff's identities:* for all $n \geq 1$,

$$\mathsf{SI(n)} \qquad \sum(-1)^{i+jt} m_{i+1+j}(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) = 0,$$

where the sum runs over all decompositions $n = i+t+j$ ($i,\ j \geq 0$ and $t \geq 1$).

The graded maps m_n for $n \geq 3$ are called *higher multiplications* of E. An A_∞-algebra E is *strictly unital* if E contains an element 1 which acts as a two-sided identity with respect to m_2, and for $n \neq 2, m_n(x_1 \otimes \cdots \otimes x_n) = 0$ if $x_i = 1$ for some i. The A_∞-algebras in this paper are always assumed to be strictly unital. An A_∞-algebra with zero m_1 is called *minimal*. An A_∞*-subalgebra* of E is a graded subspace F such that m_n maps $F^{\otimes n}$ to F for all $n \geq 1$. By an A_∞-algebra E being *generated* by E^1 we mean that for any $p \geq 2$,

$$E^p = \sum m_l(E^{i_1} \otimes \cdots \otimes E^{i_l}),$$

where the sum runs over all decompositions $i_1 + \cdots + i_l + 2 - l = p$ $(i_1, \cdots, i_l \geq 1)$ and $l \geq 1$.

Let E and F be two A_∞-algebras. A *morphism* of A_∞-algebras $f : E \to F$ is a family of graded $\mathbb{F}$-linear maps

$$f_n : E^{\otimes n} \to F, \quad n \geq 1$$

of degree $1-n$ satisfying the *Stasheff's morphism identities:* for all $n \geq 1$,

$$\mathsf{MI(n)} \qquad \sum(-1)^{i+jt} f_{i+1+j}(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) = \sum(-1)^w m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

where the first sum runs over all decompositions $n = i+t+j$ ($i,\ j \geq 0$ and $t \geq 1$), and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \cdots + i_r$ (all $i_j \geq 1$); the sign on the right-hand side is given by $w = (r-1)(i_1-1) + (r-2)(i_2-1) + \cdots + (i_{r-1}-1)$.

An A_∞-morphism in this paper is also required to be strictly unital (see [8]). A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. A morphism f is called a *strict isomorphism* if $f_i = 0$ for any $i \geq 2$ and f_1 is an isomorphism.

A_∞-algebras have been in use in topology since their introduction by Stasheff. Their applicability in an algebraic context was made clear by the minimality theorem, proven by Kadeishvili [5, 7].

Theorem 1.2. *(The minimality theorem) Let E be an A_∞-algebra. Then the cohomology H^*E has an A_∞-algebra structure such that $m_1 = 0$, m_2 is induced by m_2^E, and H^*E is quasi-isomorphic to E as A_∞-algebras.* □

The techniques used to prove the minimality theorem all yield explicit methods to compute an A_∞-algebra structure. The Ext-algebra $\underline{\mathrm{Ext}}_A^*(\mathbb{F}, \mathbb{F})$ is the cohomology of $\mathrm{End}_A(P)$, where P is any free resolution of ${}_A\mathbb{F}$. Since $E = \mathrm{End}_A(P)$ is a differential graded algebra, by the minimality theorem, $\underline{\mathrm{Ext}}_A^*(\mathbb{F}, \mathbb{F})$ has a natural A_∞-structure, which is called an A_∞*-Ext-algebra* of A. By abuse of notation we still use $E(A)$ to denote an A_∞-Ext-algebra. The importance is that the information from the A_∞-algebra $E(A)$ is sufficient to recover A.

We are mainly, in this paper, interested in the A_∞-algebra $E(A) := \bigoplus_{p,i\in\mathbb{Z}} E_i^p(A)$ which is bigraded with the lower grading inherited from the graded algebra A. Each multiplication m_n, as well as each morphism between two bigraded A_∞-algebras, must preserve the lower grading.

An A_∞-algebra $E(A)$ that we consider in the paper always comes from a free resolution. Different choice of the free resolutions yields quasi-isomorphic A_∞-algebra structures on $E(A)$. Under the assumption on A, any choice of such an A_∞-algebra structure on $E(A)$ with the multiplications $\{m_n\}_{n\geq 1}$ has the following properties: $m_1 = 0$, m_2 is the Yoneda product of $E(A)$, and $E^2(A)$ is A_∞-generated by $E^1(A)$; that is, $E_n^2(A) = m_n(E^1(A)\otimes\cdots\otimes E^1(A))$ for each $n\geq 2$. Moreover, there exists an A_∞-algebra structure on $E(A)$ such that $E(A)$ is generated by $E^1(A)$ ([6]). For more properties we refer to [6, 7] or [8, 9].

1.3. **Bi-Koszul algebras.** To extend Koszulity to a graded algebra with a bi-degree resolution of the ground field, the authors introduced what they have called bi-Koszul algebras in [10].

Definition 1.3. A *bi-Koszul algebra* (determined by $\varDelta_d$) is a connected graded algebra A whose trivial module $\mathbb{F}$ has a minimal graded free resolution $\mathcal{P}$ such that each P_n is generated in degrees $\varDelta_d(n)$ for all $n\geq 0$, where the degree distribution $\varDelta_d : \mathbb{N}\to\mathbb{N}\times\mathbb{N}$ is defined, for a fixed integer $d\geq 2$, by

$$\varDelta_d(n) = \begin{cases} \frac{n}{3}(2d,2d), & \text{if } n\equiv 0(\text{mod}3);\\ \frac{n-1}{3}(2d,2d)+(1,1), & \text{if } n\equiv 1(\text{mod}3);\\ \frac{n-2}{3}(2d,2d)+(d,d+1), & \text{if } n\equiv 2(\text{mod}3).\end{cases}$$

For simplicity, $\varDelta_d(n)$ is used to express both of its image (x,y) and of the set $\{x,y\}$. Artin-Schelter regular algebras of global dimension 4 of types (*13431*) and (*12221*) are the examples by taking $d=2$ and $d=3$, respectively. We refer to [10] for the details.

Theorem 1.4. [10] *The following statements are equivalent:*

(1) *A is a bi-Koszul algebra determined by $\varDelta_d$;*
(2) *$E(A)$ begins with $E^1(A) = E_1^1(A)$, $E^2(A) = E_d^2(A)\oplus E_{d+1}^2(A)$, $E^3(A) = E_{2d}^3(A)$, and for each $n\geq 1$,*
  (a) $E^{3n}(A) = \overbrace{E^3(A)E^3(A)\cdots E^3(A)}^{n}$,
  (b) $E^{3n+1}(A) = E^1(A)E^{3n}(A) = E^{3n}(A)E^1(A)$,
  (c) $E^{3n+2}(A) \cong E^2(A)E^{3n}(A)\oplus \mathsf{E}^2_{2nd+d+1}(J\Omega^{3n}(\mathbb{F}))$ *as $\mathbb{F}$-spaces.* □

In the above theorem, the obstruction $\mathsf{E}^2_{2nd+d+1}(J\Omega^{3n}(\mathbb{F}))$ arises from the bigger degree in $\varDelta_d(3n+2)$. We call a bi-Koszul algebra A *strongly* if the obstruction is vanished. In graded algebras setting, it is clear that the Ext-algebra $E(A)$ of a strongly bi-Koszul algebra A is generated by $E^1(A), E^2(A)$ and $E^3(A)$, but it is not sure that the Ext-algebra $E(A)$ of a bi-Koszul algebra A is finitely generated.

There is a remedy of finitely generating on $E(A)$ by using higher multiplications in Section 3.

2. A_∞-Ext-algebras of bi-Koszul algebras

In this section, we examine the possible multiplications on $E(A)$ as an A_∞-algebra for a bi-Koszul algebra A by using information about the grading of $E(A)$. An A_∞-version duality theory of bi-Koszul algebras is given. In particular, we discuss a kind of bi-Koszul algebras whose Ext-algebras are endowed with the minimal number of nonzero multiplications.

2.1. A_∞-structures on $\mathbf{E}(\mathbf{A})$.

For the sake of convenience, we write

$$m_l(E^{t_1}\cdots E^{t_l}) := m_l(E^{t_1}\otimes\cdots\otimes E^{t_l}).$$

The following lemma gives an equivalent definition of the bi-Koszul algebra which is characterized by its Ext-algebra.

Lemma 2.1. *A is a bi-Koszul algebra if and only if for any $n \geq 0$, $E^n_j(A) = 0$ for $j \notin \Delta_d(n)$.*

Proof. Similar to the proof in [1, Proposition 2.1.3]. □

Before determining all possible multiplications on $E(A)$, we claim that m_2, m_d and m_{d+1} must be non-trivial.

Proposition 2.2. *Let A be a bi-Koszul algebra determined by Δ_d. An A_∞-algebra $E := E(A)$ must have nonzero multiplications m_2, m_d and m_{d+1}.*

Proof. As mentioned in the last section, m_2 is the Yoneda product, so we need only to show that both m_d and m_{d+1} are non-trivial. Noting that $E^2 = E^2_d \oplus E^2_{d+1}$ and E^2 is generated by E^1, we have

$$m_d(E^1\cdots E^1) = E^2_d, \quad m_{d+1}(E^1\cdots E^1) = E^2_{d+1}.$$

So we get m_2, m_d and m_{d+1} are nonzero. □

One of main results of this section is

Theorem 2.3. *Let A be a bi-Koszul algebra determined by Δ_d. Then all possible non-trivial multiplications on the A_∞-Ext-algebra $E(A)$ are m_2, m_3, m_4, m_d and m_{d+1}.*

Proof. Denote $E := E(A)$. Let m_l be a multiplication on $E(A)$. Since only information about the grading of $E(A)$ is considered in the following, we can neglect the order of $E^{i_1},\cdots,E^{i_l}$ acted by m_l. Write

$$M := m_l(E^{3k_1+t_1}\cdots E^{3k_\alpha+t_\alpha}E^{3k_{\alpha+1}+2}\cdots E^{3k_l+2})$$

where $\alpha \leq l$ and $t_j = 0$ or 1 $(1 \leq j \leq \alpha)$. Denote $\beta = l-\alpha$. So

$$M \subseteq E^{3(k_1+\cdots+k_l)+(t_1+\cdots+t_\alpha)+2\beta+2-l}$$

and the lower grading of M falls into the set

$$\{2d(k_1+\cdots+k_l)+(t_1+\cdots+t_\alpha)+d\beta+j \mid j=0,1,\cdots,\beta\},$$

where $0\le t_1+\cdots+t_\alpha\le l-\beta$.

(1) If $(t_1+\cdots t_\alpha)+2\beta+2-l=3k$ $(k\ge 0)$, then

$$E^{3(k_1+\cdots+k_l)+(t_1+\cdots+t_\alpha)+2\beta+2-l}=E^{3(k_1+\cdots+k_l)+3k}_{2d(k_1+\cdots+k_l)+2dk}.$$

We have the following inequalities:

$$(t_1+\cdots+t_\alpha)+d\beta\le 2kd\le (t_1+\cdots+t_\alpha)+d\beta+\beta,$$

which produce the solutions of (k,β,l) as the following list

$$(0,0,2),\ (1,1,d),\ (1,1,d+1),\ (1,2,3),\ (2,4,4). \tag{S1}$$

(2) If $(t_1+\cdots t_\alpha)+2\beta+2-l=3k+1$ $(k\ge 0)$, then

$$E^{3(k_1+\cdots+k_l)+(t_1+\cdots+t_\alpha)+2\beta+2-l}=E^{3(k_1+\cdots+k_l)+3k+1}_{2d(k_1+\cdots+k_l)+2dk+1}.$$

The inequalities:

$$(t_1+\cdots+t_\alpha)+d\beta\le 2kd+1\le (t_1+\cdots+t_\alpha)+d\beta+\beta$$

imply the solutions of (k,β,l) in the following list

$$(0,0,2),\ (1,2,2),\ (1,2,3). \tag{S2}$$

(3) If $(t_1+\cdots t_\alpha)+2\beta+2-l=3k+2$ $(k\ge 0)$, then

$$\begin{aligned}&E^{3(k_1+\cdots+k_l)+(t_1+\cdots+t_\alpha)+2\beta+2-l}\\ =\ &E^{3(k_1+\cdots+k_l)+3k+2}_{2d(k_1+\cdots+k_l)+2dk+d}\oplus E^{3(k_1+\cdots+k_l)+3k+2}_{2d(k_1+\cdots+k_l)+2dk+d+1}.\end{aligned}$$

We have the following inequalities:

$$(t_1+\cdots+t_\alpha)+d\beta\le 2kd+d\le (t_1+\cdots+t_\alpha)+d\beta+\beta,$$

or

$$(t_1+\cdots+t_\alpha)+d\beta\le 2kd+d+1\le (t_1+\cdots+t_\alpha)+d\beta+\beta.$$

The solutions of (k,β,l) are listed in the following

$$(0,1,2),\ (1,3,3),\ (0,0,d),\ (0,1,2),\ (0,1,3),\ (0,0,d+1),\ (1,3,3),\ (1,3,4). \tag{S3}$$

In conclusion of (S1)-(S3), all possible solutions of l are $2,3,4,d$ and $d+1$. This completes the proof. □

Corollary 2.4. *Let A be a bi-Koszul algebra determined by Δ_d.*

1. *If $d=2$ or 3, the possible non-trivial multiplications on $E(A)$ are m_2, m_3 and m_4.*
2. *If $d=4$, the possible non-trivial multiplications on $E(A)$ are m_2, m_3, m_4 and m_5.*
3. *If $d\ge 5$, the possible non-trivial multiplications on $E(A)$ are m_2, m_3, m_4, m_d and m_{d+1}.* □

To describe what components the multiplications act on non-trivial, we denote

$$E^{[0]} := \bigoplus_{k\geq 0} E^{3k}, \quad E^{[1]} := \bigoplus_{k\geq 0} E^{3k+1}, \quad E^{[2]} := \bigoplus_{k\geq 0} E^{3k+2} = E^{[2]}_{(d)} \oplus E^{[2]}_{(d+1)}.$$

The following proposition is clear from the proof of Theorem 2.3.

Proposition 2.5. *Let A be a bi-Koszul algebra determined by Δ_d, E the Ext-algebra of A. Then the possible nonzero components of m_i $(i = 2, 3, 4, d, d+1)$ are:*

$*$	*fall into* $E^{[0]}$	*fall into* $E^{[1]}$	*fall into* $E^{[2]}$
m_2	$E^{[0]}E^{[0]}$	$E^{[0]}E^{[1]}$, $E^{[2]}_{(d)}E^{[2]}_{(d+1)}$	$E^{[0]}E^{[2]}$
m_3	$E^{[0]}E^{[2]}_{(d)}E^{[2]}_{(d)}$	$E^{[1]}E^{[2]}_{(d)}E^{[2]}_{(d)}$	$E^{[0]}E^{[1]}E^{[2]}_{(d)}$, $E^{[2]}E^{[2]}_{(d)}E^{[2]}_{(d)}$
m_4	$E^{[2]}_{(d)}E^{[2]}_{(d)}E^{[2]}_{(d)}E^{[2]}_{(d)}$		$E^{[1]}E^{[2]}_{(d)}E^{[2]}_{(d)}E^{[2]}_{(d)}$
m_d	$\underbrace{E^{[1]}\cdots E^{[1]}}_{d-1}E^{[2]}_{(d+1)}$		$\underbrace{E^{[1]}\cdots E^{[1]}}_{d}$
m_{d+1}	$\underbrace{E^{[1]}\cdots E^{[1]}}_{d}E^{[2]}_{(d)}$		$\underbrace{E^{[1]}\cdots E^{[1]}}_{d+1}$

$*$: *including all permutations of the components listed above.* □

Definition 2.6. We call an A_∞-algebra $E = (E; m_2, m_3, m_4, m_d, m_{d+1})$ *reduced*, if all possible nonzero components of multiplications are in the above table.

Corollary 2.7. *A is a bi-Koszul algebra if and only if any A_∞-algebra structure on $E(A)$ is reduced.*

Proof. The necessity is from Theorem 2.3 and Proposition 2.5. Now suppose that any A_∞-algebra on $E(A)$ is reduced. Take the A_∞-algebra $E(A)$ that is generated by $E^1(A)$, then A is a bi-Koszul algebra by checking the lower grading. □

2.2. **Truncated bi-Koszul algebras.** We discuss a kind of bi-Koszul algebras whose Ext-algebras are endowed with the minimal number of non-trivial multiplications m_2, m_d and m_{d+1}.

Definition 2.8. Let A be a bi-Koszul algebra determined by Δ_d, $E := E(A)$ its Ext-algebra. We say that A is *truncated* if the A_∞-Ext-algebra E only has the non-trivial multiplications m_2, m_d, m_{d+1} and the possible nonzero actions of m_i $(i = 2, d, d+1)$ are on

$*$	fall into $E^{[0]}$	fall into $E^{[1]}$	fall into $E^{[2]}$
m_2	$E^{[0]}E^{[0]}$	$E^{[0]}E^{[1]}$, $E^{[2]}_{(d)}E^{[2]}_{(d+1)}$	$E^{[0]}E^{[2]}$
m_d	$\underbrace{E^{[1]}\cdots E^{[1]}}_{d-1}E^{[2]}_{(d+1)}$		$\underbrace{E^{[1]}\cdots E^{[1]}}_{d}$
m_{d+1}	$\underbrace{E^{[1]}\cdots E^{[1]}}_{d}E^{[2]}_{(d)}$		$\underbrace{E^{[1]}\cdots E^{[1]}}_{d+1}$

$*$: including all permutations of the components listed above.

By abuse of notation we also say that the A_∞-algebra $(E; m_2, m_d, m_{d+1})$ is *truncated* in this case.

Example 2.9. All Artin-Schelter regular algebras listed in [9, Theorem A] are truncated bi-Koszul algebras.

The following result comes from Proposition 3.7 of Section 3.

Proposition 2.10. *A truncated bi-Koszul algebra must be strongly.*

A minimal A_∞-algebra is called *single* if it has only one non-trivial higher multiplication (*i.e.* a $(2,p)$-algebra discussed in [4]). Single A_∞-algebras are related to p-Koszul algebras ([4]).

A bigraded algebra is called *pure* in the sense that every component is supported in a single lower grading. We also say that a minimal A_∞-algebra which is bigraded is *pure* if the underlying bigraded algebra itself is pure.

We need the following lemma. Consider an A_∞-algebra $(E; m_2, m_d, m_t)$ ($2 < d < t$ and $2+t \neq 2d$). All non-trivial Stasheff's identities, in this case, are listed as follows:

SI(3): $m_2(m_2 \otimes 1) = m_2(1 \otimes m_2)$;
SI(d+1): $\sum_{i+j=d-1}(-1)^i m_d(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) = m_2(1 \otimes m_d) - (-1)^d m_2(m_d \otimes 1)$;
SI(t+1): $\sum_{i+j=t-1}(-1)^i m_t(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes j}) = m_2(1 \otimes m_t) - (-1)^t m_2(m_t \otimes 1)$;
SI(2d-1): $\sum_{i+j=d-1}(-1)^{i+dj} m_d(1^{\otimes i} \otimes m_d \otimes 1^{\otimes j}) = 0$;
SI(d+t-1): $\sum_{i+j=d-1}(-1)^{i+tj} m_d(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) = \sum_{i+j=t-1}(-1)^{i+dj+1} m_t(1^{\otimes i} \otimes m_d \otimes 1^{\otimes j})$;
SI(2t-1): $\sum_{i+j=t-1}(-1)^{i+tj} m_t(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}) = 0$.

Lemma 2.11. *Let E be a connected graded algebra with three graded $\mathbb{F}$-linear maps $m_n : E^{\otimes n} \to E$ $(n = 2, d, t)$. Suppose $2 < d < t$ and $2 + t \neq 2d$. Then the following statements are equivalent.*

(1) *$(E; m_2, m_d, m_t)$ is an A_∞-algebra;*
(2) *E together with $\{m_2, m_d, m_t\}$ satisfies*
   (a) *$(E; m_2, m_d)$ is single,*
   (b) *$(E; m_2, m_t)$ is single,*
   (c) *m_d and m_t obey SI(d+t-1).*

Proof. It is clear by noting that all non-trivial Stasheff's identities of a single A_∞-algebra with the higher multiplication m_p are SI(3), SI(p+1), SI(2p-1). □

From the lemma above, one may decompose a truncated A_∞-algebra into two single A_∞-algebras, while the Stasheff's identity SI(2d) serves as a bridge between two single A_∞-algebras.

Proposition 2.12. *Let A be a truncated bi-Koszul algebra determined by Δ_d ($d \geq 4$). Then both $(E; m_2, m_d)$ and $(E; m_2, m_{d+1})$ are single A_∞-algebras, generated by E^1, E^2 and E^3.*

Proof. This is a direct result of Proposition 2.10 and Lemma 2.11. □

The single A_∞-algebra $(E; m_2, m_d)$ in the proposition above is not the A_∞-Ext-algebra of any graded algebra, since there are no components acted by m_2 and m_d that fall into E^2_{d+1}; neither is $(E; m_2, m_{d+1})$.

Drawing upon the A_∞-algebras $(E; m_2, m_d)$ and $(E; m_2, m_{d+1})$, the A_∞-Ext-algebra E can be decomposed further into two single A_∞-algebras which are both pure.

Theorem 2.13. *Let A be a truncated bi-Koszul algebra determined by Δ_d ($d \geq 4$), $E := E(A)$ its Ext-algebra. Set*

$$F := E^{[0]} \oplus E^{[1]} \oplus E^{[2]}_{(d)}, \quad G := E^{[0]} \oplus E^{[1]} \oplus E^{[2]}_{(d+1)}.$$

Then

(1) *$(F; m_2, m_d)$ is a pure and single A_∞-subalgebra of $(E; m_2, m_d)$, where m_d is determined by m_2 and $m_d \mid_{(E^1)^{\otimes d}}$;*
(2) *$(G; m_2, m_{d+1})$ is a pure and single A_∞-subalgebra of $(E; m_2, m_{d+1})$, where m_{d+1} is determined by m_2 and $m_{d+1} \mid_{(E^1)^{\otimes d+1}}$.*

Proof. It is easy to justify that both $(F; m_2, m_d)$ and $(G; m_2, m_{d+1})$ are pure and single A_∞-subalgebras of $(E; m_2, m_d)$ and $(E; m_2, m_{d+1})$, respectively. And the nonzero actions of m_d are only on $(E^{[1]})^{\otimes d}$ and m_{d+1} only on $(E^{[1]})^{\otimes d+1}$.

Write $m_2(x, y)$ by xy or $x \cdot y$. Set $t = d$ or $d+1$. For any homogeneous elements $x_1, \cdots, x_t \in E^{[1]}$ and $u \in E^{[0]}$, we note that

$$m_t(x_1, \cdots, x_{i-1}, ux_i, \cdots, x_t) = m_t(x_1, \cdots, x_{i-1}u, x_i, \cdots, x_t), (2 \leq i \leq t),$$
$$m_t(ux_1, \cdots, x_t) = u \cdot m_t(x_1, \cdots, x_t),$$
$$m_t(x_1, \cdots, x_t \cdot u) = m_t(x_1, \cdots, x_t) \cdot u.$$

Since $E^{3n+j} = E^j E^{3n} = E^{3n} E^j$ ($j = 1, 2$) by Proposition 2.10, for any $x \in E^{3n+j}$ ($n \geq 0$), there exist $f, g \in E^j$ and $u, v \in E^{3n}$ such that $x = fu = vg$. If $n = 0$, set $u = v = 1$.

For any $x_1, \cdots, x_d \in E^{[1]}$, choose $z_1, \cdots, z_d \in E^1$ satisfying

$$x_d = y_d z_d, x_{d-1} y_d = y_{d-1} z_{d-1}, \cdots, x_2 y_3 = y_2 z_2, x_1 y_2 = y_1 z_1$$

where $y_1, \cdots, y_d \in E^{[0]}$. We have

$$\begin{aligned} m_d(x_1, \cdots, x_d) &= m_d(x_1, \cdots, x_{d-1}, y_d z_d) \\ &= m_d(x_1, \cdots, x_{d-1} y_d, z_d) \\ &= m_d(x_1, \cdots, y_{d-1} z_{d-1}, z_d) \\ &= \cdots \\ &= m_d(x_1 y_2, z_2, z_3 \cdots, z_d) \\ &= y_1 \cdot m_d(z_1, \cdots, z_d), \end{aligned}$$

so m_d is determined by m_2 and $m_d \mid_{(E^1)^{\otimes d}}$.

By the same method, m_{d+1} is determined by m_2 and $m_{d+1} \mid_{(E^1)^{\otimes d+1}}$. We complete the proof. □

In the A_∞-Ext-algebra $(E; \{m_i\})$ of a graded algebra A, m_2 is the Yoneda product and $m_i|_{(E^1)^{\otimes i}}$ can be computed out explicitly for every $i \geq 3$. This was demonstrated in [4]. More concretely, the single higher multiplication in either $(F; m_2, m_d)$ or $(G; m_2, m_{d+1})$ becomes definite in form.

Now, we turn to find a way in which a truncated A_∞-algebra can be formed by jointing two single A_∞-algebras together as follows.

Suppose that $(E; m_2)$ is a bigraded algebra starting with $E^1 = E^1_1$, $E^2 = E^2_d \oplus E^2_{d+1}$, $E^3 = E^3_{2d}$, and satisfying $E^{3n+i} = E^i E^{3n} = E^{3n} E^i$ for all $n \geq 1,\ i = 1, 2, 3$. Define two single A_∞-algebras $(E; m_2, m_d)$ and $(E; m_2, m_{d+1})$ such that

1. the nonzero actions of m_d are only on $(E^{[1]})^{\otimes d}$ and $E^{[1]} \cdots E^{[2]}_{(d+1)} \cdots E^{[1]}$ (including all permutations);
2. the nonzero actions of m_{d+1} are only on $(E^{[1]})^{\otimes d+1}$ and $E^{[1]} \cdots E^{[2]}_{(d)} \cdots E^{[1]}$ (including all permutations).

Then we have

Proposition 2.14. *Let $(E; m_2)$ and m_d, m_{d+1} be as above with $d \geq 4$. If m_d is compatible with m_{d+1} by* $\mathsf{SI(2d)}$ *on $(E^1)^{\otimes 2d}$. Then $(E; m_2, m_d, m_{d+1})$ is a truncated A_∞-algebra.*

Proof. We only need to show that m_d is compatible with m_{d+1} by $\mathsf{SI(2d)}$ on $(E^{[1]})^{\otimes 2d}$ by Lemma 2.11 and the nonzero actions of m_d and m_{d+1}.

Write $m_2(x, y)$ by xy or $x \cdot y$. Note the nonzero actions of m_d and m_{d+1} and the proof of Theorem 2.13. For any $x_1, \cdots, x_{2d} \in E^{[1]}$, choose $z_1, \cdots, z_{2d} \in E^1$ satisfying

$$x_{2d} = y_{2d} z_{2d},\ x_{2d-1} y_{2d} = y_{2d-1} z_{2d-1},\ \cdots,\ x_2 y_3 = y_2 z_2,\ x_1 y_2 = y_1 z_1$$

where $y_1, \cdots, y_{2d} \in E^{[0]}$. We have

$$\begin{aligned}
& m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), x_{i+d+2}, \cdots, x_{2d}) \\
= \;& m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), x_{i+d+2}, \cdots, x_{2d-1} y_{2d}, z_{2d}) \\
= \;& m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}), x_{i+d+2} y_{i+d+3}, \cdots, z_{2d}) \\
= \;& m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1}) \cdot y_{i+d+2}, z_{i+d+2}, \cdots, z_{2d}) \\
= \;& m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1}, \cdots, x_{i+d+1} y_{i+d+2}), z_{i+d+2}, \cdots, z_{2d}) \\
= \;& m_d(x_1, \cdots, x_i, m_{d+1}(x_{i+1} y_{i+2}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d}) \\
= \;& m_d(x_1, \cdots, x_i, y_{i+1} \cdot m_{d+1}(z_{i+1}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d}) \\
= \;& m_d(x_1 y_2, \cdots, z_i, m_{d+1}(z_{i+1}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d}) \\
= \;& y_1 \cdot m_d(z_1, \cdots, z_i, m_{d+1}(z_{i+1}, \cdots, z_{i+d+1}), z_{i+d+2}, \cdots, z_{2d}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& m_{d+1}(x_1, \cdots, x_i, m_d(x_{i+1}, \cdots, x_{i+d}), x_{i+d+1}, \cdots, x_{2d}) \\
= \;& y_1 \cdot m_{d+1}(z_1, \cdots, z_i, m_d(z_{i+1}, \cdots, z_{i+d}), z_{i+d+1}, \cdots, z_{2d}).
\end{aligned}$$

Set

$$\varphi := \sum_{i+j=d-1} (-1)^{i+(d+1)j} m_d(1^{\otimes i} \otimes m_{d+1} \otimes 1^{\otimes j}) + \sum_{i+j=d} (-1)^{i+dj} m_{d+1}(1^{\otimes i} \otimes m_d \otimes 1^{\otimes j}),$$

then $\varphi(x_1 \otimes \cdots \otimes x_{2d}) = y_1 \cdot \varphi(z_1 \otimes \cdots \otimes z_{2d}) = 0$ by the assumption. Therefore, $\mathsf{SI(2d)}$ holds on all $(E^{[1]})^{\otimes 2d}$.

We complete the proof. $\square$

Remark 2.1. In the proofs of Theorem 2.13 and Proposition 2.14, we ignore the $\sum$ when run up the multiplication m_2. This does not affect the results.

We finally give a condition under which the A_∞-Ext-algebra $E(A)$ is generated by $E^1(A)$.

Proposition 2.15. *Let A be a truncated bi-Koszul algebra determined by Δ_d ($d \geq 4$), $E := E(A)$ its Ext-algebra. If either $(E; m_2, m_d)$ or $(E; m_2, m_{d+1})$ is generated by E^1 and E^2. Then $(E; m_2, m_d, m_{d+1})$ is generated by E^1.*

Proof. By Proposition 2.10, E is generated by E^1, E^2, E^3 as an associative algebra. Clearly, $E^2 = m_d(\underbrace{E^1 \cdots E^1}_{d}) + m_{d+1}(\underbrace{E^1 \cdots E^1}_{d+1})$ in $(E; m_2, m_d, m_{d+1})$. Therefore, to prove the result, we need only to show that E^3 can be generated by E^1 and E^2 in $(E; m_2, m_d, m_{d+1})$, which follows either from the assumption of $(E; m_2, m_d)$ generated by E^1 and E^2 then

$$E^3 = \sum_{i=1}^{d} m_d(\underbrace{E^1 \cdots E^2_{d+1}}_{i} \cdots E^1),$$

or from the assumption of $(E; m_2, m_{d+1})$ generated by E^1 and E^2 then

$$E^3 = \sum_{i=1}^{d+1} m_{d+1}(\underbrace{E^1 \cdots E^2_{d}}_{i} \cdots E^1).$$

We get the result. $\square$

3. Balanced generating

For a bi-Koszul algebra A, it is a question whether $E(A)$ is finitely generated as a graded algebra. In this section, we generalize the concept of "generating", and show that $E(A)$ is $[m_2, m_3]$-finitely generated by $E^1(A), E^2(A)$ and $E^3(A)$ for any bi-Koszul algebra A. An equivalent statement of a bi-Koszul algebra is given in terms of such concept.

3.1. $[\mathbf{m_2}, \mathbf{m_3}]$**-Generating.** The original concept of "generating" in the associative algebra setting is defined with respect to the multiplication. When we work in the field of A_∞-algebras, we need a generalized concept of "generating" to reflect certain balance between multiplications and elements.

Definition 3.1. Let E be an A_∞-algebra. Suppose there exists a fixed integer l and multiplications $m_{n_1}, \cdots, m_{n_t}$ such that, for any $p > l$,

$$E^p = \sum_{\substack{k_1+\cdots+k_{n_i}+2-n_i=p \\ k_1,\cdots,k_{n_i}\geq 1;\ 1\leq i\leq t}} m_{n_i}(E^{k_1} \otimes \cdots \otimes E^{k_{n_i}}).$$

We say that E is $[m_{n_1},\cdots,m_{n_t}]$-*finitely generated by* $E^1,\cdots,E^l$.

Remark 3.1. In the case of $m_1=0$ and $[m_{n_1},\cdots,m_{n_t}]=[m_2]$, the concept is the original one of finitely generating as an associative graded algebra.

Here are the examples.

If A is a p-Koszul algebra ($p\geq 3$), then any A_∞-algebra $(E(A);m_2,m_p)$ is $[m_2,m_p]$-finitely generated by $E^1(A)$. This is obtained by noting the facts that $E(A)$ is generated by $E^1(A)$ and $E^2(A)$, while $E^2(A)=m_p(E^1(A)\otimes\cdots\otimes E^1(A))$ ([4, Theorem 2.5]).

If A is a bi-Koszul algebra, then there exists an A_∞-algebra $(E(A);m_2,m_3,m_4,m_d,m_{d+1})$ such that $E(A)$ is $[m_2,m_3,m_4,m_d,m_{d+1}]$-finitely generated by $E^1(A)$.

If we admit the set of multiplications to be infinite, Keller's result tells that there exists an A_∞-algebra structure on $E(A)$ which is generated by $E^1(A)$ [6, Proposition 1(b)].

It is natural to expect that the multiplications in the set $\{m_{n_1},\cdots,m_{n_t}\}$ as less as possible. When A is a bi-Koszul algebra, though we can not claim $E(A)$ is $[m_2]$-finitely generated, it does be finitely generated as long as to add one higher multiplication.

Now let A be a bi-Koszul algebra determined by $\varDelta_d$, and $E:=E(A)$ the Ext-algebra of A in the following. Denote

- $\mathbf{U}^{3k+2}$: the sum of the actions of m_3 on all permutations of E^{3k_1}, E^{3k_2+1} and $E^{3k_3+2}_{2dk_3+d}$ for any $k_1\geq 1, k_2,k_3\geq 0$ with $k_1+k_2+k_3=k$;
- $\mathbf{V}^{3k+2}$: the sum of the actions of m_3 on all permutations of $E^{3k_1+2}_{2dk_1+d}$, $E^{3k_2+2}_{2dk_2+d}$ and $E^{3k_3+2}_{2dk_3+d+1}$ for any $k_1,k_2,k_3\geq 0$ with $k_1+k_2+k_3=k-1$;
- $\mathbf{W}^{3k+2}$: the sum of the actions of m_4 on all permutations of E^{3k_1+1}, $E^{3k_2+2}_{2dk_2+d}$, $E^{3k_3+2}_{2dk_3+d}$ and $E^{3k_4+2}_{2dk_4+d}$ for any $k_1,k_2,k_3,k_4\geq 0$ with $k_1+k_2+k_3+k_4=k-1$.

Proposition 3.2. *Let A be a bi-Koszul algebra. Then for any $k\geq 1$,*

$$E^{3k+3}=m_2(E^3E^{3k})=m_2(E^{3k}E^3);$$
$$E^{3k+1}=m_2(E^1E^{3k})=m_2(E^{3k}E^1);$$
$$E^{3k+2}_{2dk+d}=m_2(E^2_dE^{3k})=m_2(E^{3k}E^2_d).$$

Proof. By Theorem 1.4, we need only to show the last equality:

$$m_2(E^2_dE^{3k})=m_2(E^{3k}E^2_d).$$

Since $m_2(E^2_dE^{3k})=m_2(m_d(E^1\cdots E^1)E^{3k})$, to get $m_2(E^2_dE^{3k})\subseteq m_2(E^{3k}E^2_d)$ we need only to verify $m_2(m_d(x_1,\cdots,x_d),y)\in m_2(E^{3k}E^2_d)$ for any $x_1,\cdots,x_d\in E^1$ and $y\in E^{3k}$. This is performed by using the Stasheff's identity $\mathsf{SI(d+1)}$

$$m_2(m_d(x_1,\cdots,x_d),y)=m_d(x_1,\cdots,x_{d-1},m_2(x_d,y))$$

with $m_2(x_d,y)\in E^{3k+1}$. Since $E^{3k+1}=m_2(E^{3k}E^1)$,

$$\begin{aligned} m_d(x_1,\cdots,x_{d-1},m_2(x_d,y)) &= m_d(x_1,\cdots,x_{d-1},m_2(y',x'_d)) \\ &= m_d(x_1,\cdots,m_2(x_{d-1},y'),x'_d) \end{aligned}$$

with $y' \in E^{3k}$ and $x'_d \in E^1$. We can continue the foregoing procedure to obtain

$$\begin{aligned} m_d(x_1, \cdots, x_{d-1}, m_2(x_d, y)) &= m_d(m_2(z, x'_1), \cdots, x'_d) \\ &= m_2(z, m_d(x'_1, \cdots, x'_d)) \end{aligned}$$

with $z \in E^{3k}$ and $x'_1, \cdots, x'_d \in E^1$. Thus, $m_2(E^2_d E^{3k}) \subseteq m_2(E^{3k} E^2_d)$.

The converse $m_2(E^{3k} E^2_d) \subseteq E^{3k+2}_{2dk+d} = m_2(E^2_d E^{3k})$ is clear from Theorem 1.4 again. □

Lemma 3.3. *Let A be a bi-Koszul algebra. Assume $k_i \geq 0$ $(i = 1, \cdots, d+1)$ and $k_1 + \cdots + k_{d+1} = k \geq 1$. Then*

$$m_{d+1}(E^{3k_1+1} \cdots E^{3k_{d+1}+1}) \subseteq \sum_{\substack{i_1+i_2=k, \\ i_1 \geq 1, i_2 \geq 0}} m_2(E^{3i_1} E^{3i_2+2}) + \mathbf{U}^{3k+2}.$$

Proof. There exists an integer i $(1 \leq i \leq d+1)$ such that $k_i \geq 1$. If $2 \leq i \leq d+1$, using $\mathsf{SI(d+2)}$, we get

$$\begin{aligned} & m_{d+1}(E^{3k_1+1} \cdots E^{3k_{d+1}+1}) \\ = \; & m_{d+1}(E^{3k_1+1} \cdots m_2(E^{3k_i} E^1) \cdots E^{3k_{d+1}+1}) \\ \subseteq \; & m_{d+1}(E^{3k_1+1} \cdots E^{3k_{i-1}+3k_i+1} E^1 \cdots E^{3k_{d+1}+1}) + \mathbf{U}^{3k+2} \\ \subseteq \; & m_{d+1}(E^{3(k_1+\cdots+k_i)+1} \cdots E^1 E^1 \cdots E^{3k_{d+1}+1}) + \mathbf{U}^{3k+2} \end{aligned}$$

with $k_1 + \cdots + k_i \geq 1$.

So we can assume $k_1 \geq 1$. Using $\mathsf{SI(d+2)}$ again,

$$\begin{aligned} & m_{d+1}(E^{3k_1+1} \cdots E^{3k_{d+1}+1}) \\ = \; & m_{d+1}(m_2(E^{3k_1} E^1) E^{3k_2+1} \cdots E^{3k_{d+1}+1}) \\ \subseteq \; & m_3(E^{3k_1} E^{3k_2+\cdots+3k_d+2} E^{3k_{d+1}+1}) + m_3(E^{3k_1} E^1 E^{3k_2+\cdots+3k_{d+1}+2}) \\ & + m_2(E^{3k_1} E^{3k_2+\cdots+3k_{d+1}+2}). \end{aligned}$$

We complete the proof. □

Lemma 3.4. *Let A be a bi-Koszul algebra. Then*

$$\mathbf{W}^{3k+2} \subseteq \sum_{\substack{i_1+i_2=k, \\ i_1 \geq 1, i_2 \geq 0}} m_2(E^{3i_1} E^{3i_2+2}) + \mathbf{U}^{3k+2} + \mathbf{V}^{3k+2}.$$

Proof. By ignoring the lower grading, we may write

$$W^{3k+2} = \sum m_4(E^{3k_1+i_1} E^{3k_2+i_2} E^{3k_3+i_3} E^{3k_4+i_4})$$

where the sum runs over all $i_1 + i_2 + i_3 + i_4 = 7$ $(i_j = 1 \text{ or } 2)$ and $k_j \geq 0$.

First, assume $k_1 + k_2 + k_3 + k_4 = 0$. In this case, the first or last component, say the last component, must be E^2_d. Using $\mathsf{SI(d+3)}$, we have

$$\begin{aligned} & m_4(E^{i_1} E^{i_2} E^{i_3} E^2_d) \\ = \; & m_4(E^{i_1} E^{i_2} E^{i_3} m_d(E^1 \cdots E^1)) \\ \subseteq \; & m_{d+1}(E^4 E^1 \cdots E^1) + m_{d+1}(E^1 E^4 \cdots E^1) + \mathbf{U}^{3k+2}. \end{aligned}$$

Next, consider $k_1+k_2+k_3+k_4 \geq 1$.

If $k_2 \geq 1$,

$$\begin{aligned} & m_4(E^{3k_1+i_1}E^{3k_2+i_2}E^{3k_3+i_3}E^{3k_4+i_4}) \\ \subseteq\ & m_4(E^{3k_1+3k_2+i_1}E^{i_2}E^{3k_3+i_3}E^{3k_4+i_4}) + \mathbf{U}^{3k+2} + \mathbf{V}^{3k+2}. \end{aligned}$$

If $k_3 \geq 1$, by the similar method we get

$$\begin{aligned} & m_4(E^{3k_1+i_1}E^{3k_2+i_2}E^{3k_3+i_3}E^{3k_4+i_4}) \\ \subseteq\ & m_4(E^{3k_1+3k_2+3k_3+i_1}E^{i_2}E^{i_3}E^{3k_4+i_4}) + \mathbf{U}^{3k+2} + \mathbf{V}^{3k+2}. \end{aligned}$$

Now we assume that $k_1 \geq 1$ (the case of $k_4 \geq 1$ is symmetrical). Whether $E^{3k_1+i_1} = E^{3k_1+1}$ or E^{3k_1+2}, by $\mathsf{SI(5)}$ we get

$$m_4(E^{3k_1+i_1}E^{3k_2+i_2}E^{3k_3+i_3}E^{3k_4+i_4}) \subseteq \mathbf{U}^{3k+2} + \mathbf{V}^{3k+2}.$$

We complete the proof. □

Examining the table in Proposition 2.5 again, the following result is clear from Proposition 3.2, the lemmas 3.3 and 3.4.

Corollary 3.5. *Let A be a bi-Koszul algebra. Then the actions of m_4, m_d, m_{d+1} which fall into $E^{\geq 4}$ are determined by the actions of m_2 and m_3.* □

Now we can state an equivalent statement of the bi-Koszul algebra.

Theorem 3.6. *Let A be a locally finite, connected graded algebra generated in degree 1. Then A is a bi-Koszul algebra if and only if there exists a reduced A_∞-algebra $(E(A);\{m_i\})$ which is $[m_2, m_3]$-finitely generated by $E^1(A)$, $E^2(A)$ and $E^3(A)$ with $E^1(A) = E^1_1(A)$, $E^2(A) = E^2_d(A) \oplus E^2_{d+1}(A)$, $E^3(A) = E^3_{2d}(A)$.*

Proof. Assume that A is a bi-Koszul algebra. Take an A_∞-algebra $(E(A);\{m_i\})$ that is generated by $E^1(A)$, then $(E(A);\{m_i\})$ is a reduced A_∞-algebra by Corollary 2.7. By Proposition 3.2, the lemmas 3.3 and 3.4, we get $E(A)$ begins with $E^1(A) = E^1_1(A)$, $E^2(A) = E^2_d(A) \oplus E^2_{d+1}(A)$, $E^3(A) = E^3_{2d}(A)$, and for each $k \geq 1$,

$$\begin{aligned} E^{3k+3} &= \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2}), \\ E^{3k+1} &= \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2+1}) = \sum_{k_1+k_2=k} m_2(E^{3k_1+1}E^{3k_2}), \\ E^{3k+2} &= \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2+2}) + \sum_{k_1+k_2=k} m_2(E^{3k_1+2}E^{3k_2}) + \mathbf{U}^{3k+2} + \mathbf{V}^{3k+2}, \end{aligned}$$

which implies that $(E(A);\{m_i\})$ is $[m_2, m_3]$-finitely generated by $E^1(A), E^2(A)$ and $E^3(A)$.

The converse is straightforward by comparing the lower grading. □

The result above tells that the obstruction in Theorem 1.4 can be described by the multiplications m_2 and m_3. Using the multiplications m_2 and m_3, we may also give a criteria for a bi-Koszul algebra to be strongly.

Proposition 3.7. *Let A be a bi-Koszul algebra. Then A is strongly if and only if for any A_∞-algebra $(E(A);\{m_i\})$,*

$$\mathbf{U}^{3k+2} \subseteq m_2(E^2_{d+1}E^{3k}), \quad \textit{and}$$
$$\mathbf{V}^{3k+2} \subseteq \sum_{k_1+k_2=k} m_2(E^{3k_1}E^{3k_2+2}) + \sum_{k_1+k_2=k} m_2(E^{3k_1+2}E^{3k_2}).$$

Proof. The necessity is obvious. To show the condition being sufficient, we need only to check

$$E^{3k+2}_{2dk+d+1} = m_2(E^2_{d+1}E^{3k}), \quad \text{for any } \ k \geq 1. \tag{$*$}$$

The reason is that the obstruction arises only from the bigger degree $2dk+d+1$ in $\Delta_d(3k+2)$ as we pointed before.

We first show that

$$m_2(E^2_{d+1}E^{3k}) + \mathbf{U}^{3k+2} = m_2(E^{3k}E^2_{d+1}) + \mathbf{U}^{3k+2}. \tag{$**$}$$

In fact, by the Stasheff's identity $\mathsf{SI(d+2)}$

$$\begin{aligned} m_2(E^{3k}E^2_{d+1}) &= m_2(E^{3k}m_{d+1}(E^1\cdots E^1)) \\ &\subseteq m_{d+1}(m_2(E^{3k}E^1)\cdots E^1) + \mathbf{U}^{3k+2} \\ &= m_{d+1}(m_2(E^1E^{3k})\cdots E^1) + \mathbf{U}^{3k+2} \\ &\subseteq m_{d+1}(E^1m_2(E^{3k}E^1)\cdots E^1) + \mathbf{U}^{3k+2} \\ &\subseteq \cdots\cdots \\ &\subseteq m_2(E^2_{d+1}E^{3k}) + \mathbf{U}^{3k+2}, \end{aligned}$$

which implies one inclusion relation, the opposite inclusion is similar to prove.

Using $(**)$ and the condition on $\mathbf{U}^{3k+2}$, we have $m_2(E^{3k}E^2_{d+1}) \subseteq m_2(E^2_{d+1}E^{3k})$. Again from the conditions on $\mathbf{U}^{3k+2}$, $\mathbf{V}^{3k+2}$ and the proof of Theorem 3.6, we get

$$E^{3k+2} = \sum m_2(E^{3k_1}E^{3k_2+2}) + \sum m_2(E^{3k_1+2}E^{3k_2}).$$

We prove $(*)$ by induction on $k \geq 1$.

$$E^5_{3d+1} = m_2(E^3E^2_{d+1}) + m_2(E^2_{d+1}E^3) = m_2(E^2_{d+1}E^3).$$

Suppose that $E^{3i+2}_{2di+d+1} = m_2(E^2_{d+1}E^{3i})$ for all $1 \leq i < k$. Now, for any $k_1+k_2 = k \geq 2$ ($k_1 \geq 1$ and $k_2 \geq 1$),

$$m_2(E^{3k_1}E^{3k_2+2}_{2dk_2+d+1}) = m_2(E^{3k_1}m_2(E^2_{d+1}E^{3k_2})) \subseteq m_2(E^{3k_1+2}_{2dk_1+d+1}E^{3k_2})$$

and

$$m_2(E^{3k_1+2}_{2dk_1+d+1}E^{3k_2}) = m_2(m_2(E^2_{d+1}E^{3k_1})E^{3k_2}) \subseteq m_2(E^2_{d+1}E^{3k}).$$

This proves, for any $k \geq 1$, $E^{3k+2}_{2dk+d+1} \subseteq m_2(E^2_{d+1}E^{3k})$, the opposite inclusion is trivial, so we have $(*)$.

Hence the bi-Koszul algebra A is strongly. $\square$

Corollary 3.8. *If A is a bi-Koszul algebra with $gl.dim(A) \leq 4$, or its Ext-algebra $E(A)$ with $m_3 = 0$, then A is a strongly bi-Koszul algebra.*

Proof. It is clear since each assumption implies $\mathbf{U}^{3k+2} = \mathbf{V}^{3k+2} = 0$ for any $k > 0$ by Proposition 3.7. □

Example 3.9. Any truncated bi-Koszul algebra is strongly.

3.2. **Generated by $\mathbf{E^1(A)}$.** Let A be a bi-Koszul algebra, Theorem 3.6 tells that any A_∞-algebra $E(A)$ is $[m_2, m_3]$-finitely generated by $E^1(A)$, $E^2(A)$ and $E^3(A)$. On the other hand, Keller has claimed that there exists an A_∞-algebra structure on $E(A)$ which is generated by $E^1(A)$. In this subsection, we discuss the universality of the property that $E(A)$ is generated by $E^1(A)$ as an A_∞-algebra.

For example, let A be an Artin-Schelter regular algebra listed in [9, Theorem A], then any A_∞-algebra $E(A)$ is generated by $E^1(A)$.

Before discussing, we give a general result which points out that a strict isomorphism of A_∞-algebras can be obtained from a quasi-isomorphism of A_∞-algebras. This was found in [4] for single A_∞-algebras.

Lemma 3.10. *Let $(E;\{m_i\})$ and $(E';\{m'_i\})$ be two minimal A_∞-algebras, and $\{f_i\} : (E;\{m_i\}) \to (E';\{m'_i\})$ a quasi-isomorphism between them. Then*

(1) *$(E';\{m''_i\})$ is a minimal A_∞-algebra where $m''_i := f_1 m_i(f_1^{-1} \otimes \cdots \otimes f_1^{-1})$ with $m''_2 = m'_2$.*
(2) *$\{g_i\} : (E;\{m_i\}) \to (E';\{m''_i\})$ is a strict isomorphism of A_∞-algebras where $g_1 = f_1$ and $g_i = 0$ for all $i \geq 2$.*

Proof. Since $m_1 = m'_1 = 0$, $f_1 : (E; m_2) \to (E'; m'_2)$ is an isomorphism. To prove the first statement, we need the Stasheff's morphism identities $\mathsf{SI(n)}$ $(n = 1, 2, \cdots)$ for $\{m''_i\}$. Note that the degrees of both f_1 and f_1^{-1} are zero, the Koszul sign convention can be neglected in the following. For any $i+t+j = n$ and $l = i+1+j$,

$$\begin{aligned}
& m''_l(1^{\otimes i} \otimes m''_t \otimes 1^{\otimes j}) \\
= \;& f_1 m_l(f_1^{-1} \otimes \cdots \otimes f_1^{-1})(1^{\otimes i} \otimes f_1 m_t(f_1^{-1} \otimes \cdots \otimes f_1^{-1}) \otimes 1^{\otimes j}) \\
= \;& f_1 m_l(f_1^{-1} \otimes \cdots \otimes f_1^{-1})(1^{\otimes i} \otimes f_1 m_t \otimes 1^{\otimes j})(1^{\otimes i} \otimes f_1^{-1} \otimes \cdots \otimes f_1^{-1} \otimes 1^{\otimes j}) \\
= \;& f_1 m_l(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j})(f_1^{-1} \otimes \cdots \otimes f_1^{-1}),
\end{aligned}$$

hence

$$\begin{aligned}
& \sum_{n=i+t+j} (-1)^{i+tj} m''_l(1^{\otimes i} \otimes m''_t \otimes 1^{\otimes j}) \\
= \;& \sum_{n=i+t+j} (-1)^{i+tj} f_1(m_l(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}))(f_1^{-1} \otimes \cdots \otimes f_1^{-1}) \\
= \;& f_1\Big(\sum_{n=i+t+j} (-1)^{i+tj}(m_l(1^{\otimes i} \otimes m_t \otimes 1^{\otimes j}))\Big)(f_1^{-1} \otimes \cdots \otimes f_1^{-1}) \\
= \;& 0.
\end{aligned}$$

Moreover, $m''_2 = f_1 m_2(f_1^{-1} \otimes f_1^{-1}) = m'_2(f_1 \otimes f_1)(f_1^{-1} \otimes f_1^{-1}) = m'_2$.

Clearly, $f_1 m_i = m''_i(f_1 \otimes \cdots \otimes f_1)$. So $\{g_i\}$ is a strict isomorphism between $(E;\{m_i\})$ and $(E';\{m''_i\})$. □

Let A be a bi-Koszul algebra determined by Δ_d, $E := E(A)$ the Ext-algebra of A. There is a quasi-isomorphism between two A_∞-algebra structures on $E(A)$:

$$\{f_i\} : (E(A); \{m_i\}) \to (E(A); \{m'_i\}).$$

Now assume that $(E(A); \{m_i\})$ is generated by $E^1(A)$, it is a natural question that whether the same claim is true for $(E(A); \{m'_i\})$.

The following facts are immediately:

(i) if $n \geq 2$, $f_n(E^1 \cdots E^1) = 0$;
(ii) if $d \geq 3$, $m_2(E^1E^1) = m_2(E^1E^2) = m_2(E^2E^1) = 0$;
(iii) there are only m'_d and m'_{d+1} whose actions can fall into E^3 in $(E(A); \{m'_i\})$;
(iv) $f_1 : E(A) \to E(A)$ is an isomorphism.

Combining with Lemma 2.1 and Proposition 2.5, we can write the Stasheff's morphism identities, for small n or in some special cases, more clearly.

(1) $\mathsf{MI(2)}$: $f_1m_2 = m'_2(f_1 \otimes f_1)$;
(2) $\mathsf{MI(3)}$: $f_1m_3 + f_2(m_2 \otimes 1) - f_2(1 \otimes m_2)$
$= m'_3(f_1 \otimes f_1 \otimes f_1) + m'_2(f_1 \otimes f_2) - m'_2(f_2 \otimes f_1)$;
(3) $\mathsf{MI(d)}$ acting on $E^1 \cdots E^1$ or $E^1 \cdots E^2_{d+1} \cdots E^1$ can be reduced as

$$f_1m_d = m'_d(f_1 \otimes \cdots \otimes f_1);$$

(4) $\mathsf{MI(d+1)}$ acting on $E^1 \cdots E^1$ can be reduced as

$$f_1m_{d+1} + (-1)^d f_2(m_d \otimes 1) + (-1)^{d+1} f_2(1 \otimes m_d) = m'_{d+1}(f_1 \otimes \cdots \otimes f_1);$$

(5) $\mathsf{MI(d+1)}$ acting on $E^1 \cdots E^2_d \cdots E^1$ can be reduced as

$$\begin{aligned} &f_1m_{d+1} + (-1)^d f_2(m_d \otimes 1) + (-1)^{d+1} f_2(1 \otimes m_d) \\ = {}& m'_{d+1}(f_1 \otimes \cdots \otimes f_1) + \textstyle\sum_{1\leq j\leq d} (-1)^{d-j} m'_d(\underbrace{f_1 \otimes \cdots \otimes f_2}_{j} \otimes \cdots \otimes f_1). \end{aligned}$$

Proposition 3.11. *Assume $E^3(A)$ is generated by $E^1(A)$ and $E^2(A)$ with $\{m_i\}$. If $f_2(m_d \otimes 1) = f_2(1 \otimes m_d)$, then $E^3(A)$ is also generated by $E^1(A)$ and $E^2(A)$ with $\{m'_i\}$.*

Proof. The hypothesis on $E^3(A)$ tells us that

$$E^3 = \sum m_d(E^1 \cdots E^2_{d+1} \cdots E^1) + \sum m_{d+1}(E^1 \cdots E^2_d \cdots E^1).$$

So $E^3 = f_1(E^3) = \sum f_1m_d(E^1 \cdots E^2_{d+1} \cdots E^1) + \sum f_1m_{d+1}(E^1 \cdots E^2_d \cdots E^1)$.

By the Stasheff's morphism identities listed above, we have

$$\begin{aligned} & f_1m_d(E^1 \cdots E^2_{d+1} \cdots E^1) \\ \subseteq \quad & m'_d(f_1 \otimes \cdots \otimes f_1)(E^1 \otimes \cdots \otimes E^2_{d+1} \otimes \cdots \otimes E^1) \\ = \quad & m'_d(E^1 \cdots E^2_{d+1} \cdots E^1), \end{aligned}$$

and

$$\begin{array}{rl} & f_1 m_{d+1}(E^1 \cdots E_d^2 \cdots E^1) \\ \subseteq & m'_{d+1}(f_1 \otimes \cdots \otimes f_1)(E^1 \otimes \cdots \otimes E_d^2 \otimes \cdots \otimes E^1) \\ & + \sum\limits_{1 \le j \le d} (-1)^{d-j} m'_d(f_1 \otimes \cdots \otimes f_2 \otimes \cdots \otimes f_1)(E^1 \otimes \cdots \otimes E_d^2 \otimes \cdots \otimes E^1). \\ \subseteq & m'_{d+1}(E^1 \cdots E_d^2 \cdots E^1) + \sum m'_d(E^1 \cdots E_{d+1}^2 \cdots E^1). \end{array}$$

Thus, $E^3 \subseteq \sum m'_d(E^1 \cdots E_{d+1}^2 \cdots E^1) + \sum m'_{d+1}(E^1 \cdots E_d^2 \cdots E^1)$. We complete the proof. $\square$

Now, we can prove the main theorem of this subsection.

Theorem 3.12. *Let $\{f_i\} : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$ be a quasi-isomorphism with $f_2(m_i \otimes 1) = f_2(1 \otimes m_i)$ for $i = 2, d$, suppose that $(E(A); \{m_i\})$ is generated by $E^1(A)$. Then $(E(A); \{m'_i\})$ is also generated by $E^1(A)$.*

Proof. Since $m_1 = m'_1 = 0$, $f_1 : (E(A), m_2) \to (E(A), m'_2)$ is an isomorphism with degree zero. By the proof of Theorem 3.6 and $\mathsf{MI(2)}$, we have

$$\begin{array}{rl} E^{3k+3} & = f_1(E^{3k+3}) = f_1(\sum m_2(E^{3k_1}E^{3k_2})) \\ & \subseteq \sum m'_2(f_1 \otimes f_1)(E^{3k_1} \otimes E^{3k_2}) = \sum m'_2(E^{3k_1}E^{3k_2}); \\ E^{3k+1} & = f_1(E^{3k+1}) = f_1(\sum m_2(E^{3k_1+1}E^{3k_2})) \\ & \subseteq \sum m'_2(f_1 \otimes f_1)(E^{3k_1+1} \otimes E^{3k_2}) = \sum m'_2(E^{3k_1+1}E^{3k_2}); \\ E^{3k+2} & = f_1(E^{3k+2}) \\ & = f_1(\sum m_2(E^{3k_1}E^{3k_2+2}) + \sum m_2(E^{3k_1+2}E^{3k_2}) + \mathbf{U}^{3k+2} + \mathbf{V}^{3k+2}) \\ & \subseteq \sum m'_2(E^{3k_1}E^{3k_2+2}) + \sum m'_2(E^{3k_1+2}E^{3k_2}) + f_1(\mathbf{U}^{3k+2}) + f_1(\mathbf{V}^{3k+2}). \end{array}$$

For any $E^{3i_1}E^{3i_2+1}E^{3i_3+2}_{2di_3+d}$ $(i_1 + i_2 + i_3 = k, i_1 \ge 1, i_2, i_3 \ge 0)$, the assumption $f_2(m_2 \otimes 1) = f_2(1 \otimes m_2)$ and $\mathsf{MI(3)}$ imply that

$$\begin{array}{rl} & f_1 m_3(E^{3i_1}E^{3i_2+1}E^{3i_3+2}_{2di_3+d}) \\ \subseteq & m'_3(E^{3i_1}E^{3i_2+1}E^{3i_3+2}_{2di_3+d}) + m'_2(f_2(E^{3i_1}E^{3i_2+1})f_1(E^{3i_3+2}_{2di_3+d})) \\ & +m'_2(f_1(E^{3i_1})f_2(E^{3i_2+1}E^{3i_3+2}_{2di_3+d})) \\ \subseteq & m'_3(E^{3i_1}E^{3i_2+1}E^{3i_3+2}_{2di_3+d}) + m'_2(E^{3i_1+3i_2}E^{3i_3+2}_{2di_3+d}) \\ & +m'_2(E^{3i_1}E^{3i_2+3i_3+2}_{2di_2+2di_3+d}). \end{array}$$

By the same method, we obtain that the action of f_1 on every component of $\mathbf{U}^{3k+2}$ or $\mathbf{V}^{3k+2}$ falls into $\sum m'_2(E^{3k_1}E^{3k_2+2}) + \sum m'_2(E^{3k_1+2}E^{3k_2}) + \mathbf{U}'^{3k+2} + \mathbf{V}'^{3k+2}$ where $\mathbf{U}'^{3k+2}$ and $\mathbf{V}'^{3k+2}$ correspond to $\mathbf{U}^{3k+2}$ and $\mathbf{V}^{3k+2}$, respectively, changing the multiplication m_3 to m'_3. Thus,

$$E^{3k+2} = \sum m'_2(E^{3k_1}E^{3k_2+2}) + \sum m'_2(E^{3k_1+2}E^{3k_2}) + \mathbf{U}'^{3k+2} + \mathbf{V}'^{3k+2}.$$

Since E^2 is generated by E^1, and E^3 is generated by E^1 and E^2 by Proposition 3.11, we get $(E(A); \{m'_i\})$ is also generated by E^1. $\square$

Corollary 3.13. *Assume the bi-Koszul algebra A is strongly. If $(E(A); \{m_i\})$ is generated by E^1, and $f_2(m_d \otimes 1) = f_2(1 \otimes m_d)$, then $(E(A); \{m'_i\})$ is also generated by E^1.* □

Continuing to consider the quasi-isomorphism between two A_∞-structures on $E(A)$, $\{f_i\} : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$, some extra hypothesis will make $\{f_i\}$ to be a strict isomorphism which guarantees the properties of such two A_∞-algebras identify with each other.

Theorem 3.14. *If $f_2(m_i \otimes 1) = f_2(1 \otimes m_i)$ for $i = 2, d$, and*

$$\sum_{1 \leq j \leq i} (-1)^{i-j} m'_i(\underbrace{f_1 \otimes \cdots \otimes f_2}_{j} \otimes \cdots f_1) = 0,$$

then $g = f_1 : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$ is a strict isomorphism.

Proof. By Lemma 3.10, $g = f_1 : (E(A); \{m_i\}) \to (E(A); \{m''_i\})$ is a strict isomorphism and $m''_2 = m'_2 = m_2$. The assumption directly implies $m''_3 = f_1 m_3 (f_1^{-1} \otimes f_1^{-1} \otimes f_1^{-1}) = m'_3$. For any $x_1, \cdots, x_d \in E^1$, or one of x_j's in E^2_{d+1} and the others in E^1, we have

$$m''_d(x_1, \cdots, x_d) = f_1 m_d (f_1^{-1}(x_1), \cdots, f_1^{-1}(x_d)) = m'_d(x_1, \cdots, x_d)$$

by the Stasheff's morphism identities listed above. So $m''_d|_{(E^1)^{\otimes d}} = m'_d|_{(E^1)^{\otimes d}}$ and $m''_d|_{E^1 \cdots E^2_{d+1} \cdots E^1} = m'_d|_{E^1 \cdots E^2_{d+1} \cdots E^1}$. By the same method, we check that $m''_{d+1}|_{(E^1)^{\otimes(d+1)}} = m'_{d+1}|_{(E^1)^{\otimes(d+1)}}$ and $m''_{d+1}|_{E^1 \cdots E^2_d \cdots E^1} = m'_{d+1}|_{E^1 \cdots E^2_d \cdots E^1}$. Thus, $(E(A); \{m'_i\})$ and $(E(A); \{m''_i\})$ are the same. The result follows immediately. □

Corollary 3.15. *Assume the bi-Koszul algebra A is strongly. If $f_2(m_d \otimes 1) = f_2(1 \otimes m_d)$ and $\sum_{1 \leq j \leq d} (-1)^{d-j} m'_d(\underbrace{f_1 \otimes \cdots \otimes f_2}_{j} \otimes \cdots \otimes f_1) = 0$. Then*

$$g = f_1 : (E(A); \{m_i\}) \to (E(A); \{m'_i\})$$

is a strict isomorphism.

Proof. By the proof of Theorem 3.14. □

References

1. A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–52
2. R. Berger, *Koszulity of nonquadratic algebras*, J. Algebra, **239** (2001), 705-734.
3. T. Cassidy and B. Shelton, *Generalizing the notion of a Koszul algebra*, Math. Z., **260** (2008), 93-114.
4. J.-W. He and D.-M. Lu, *Higher Koszul algebras and A-infinity algebras*, J. Algebra, **293** (2005), 335-362.
5. T. V. Kadeishvili, *On the theory of homology of fiber spaces*, (Russian) International Topology Conference (Moscow State Univ., Moscow, 1979). Uspekhi Mat. Nauk **35** (1980), no. 3 (213), 183–188. The English translation was published in Russian Math. Surveys, **35**, no.3 (1980), 231-238.

6. B. Keller, *A-infinity algebras in representation theory*, Contribution to the Proceedings of ICRA IX. Beijing: Peking University Press, 2000.
7. B. Keller, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl., **3** (2001), 1-35 (electronic).
8. D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, *A-infinity algebras for ring theorists*, Algebra Colloq., **11** (2004), 91-128.
9. D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, *Regular algebras of dimension 4 and their A_∞-Ext-algebras*, Duke Math. J., **137** (2007), 537-584.
10. D.-M. Lu and J.-R. Si, *Koszulity of algebras with non-pure resolutions*, Comm. Alg., accepted for publication.
11. J.-F. Lü, J.-W. He and D.-M. Lu, *Piecewise-Koszul algebras*, Sci. China Ser. A, **50** (2007), 1795-1804.
12. S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc., **152** (1970), 39-60.

(Si) Department of Mathematics, Zhejiang University, Hangzhou 310027, China
E-mail address: `sijunru@126.com`

(Lu) Department of Mathematics, Zhejiang University, Hangzhou 310027, China
E-mail address: `dmlu@zju.edu.cn`